ADVANCES IN MATHEMATICS 77, 137-155 (1989)

Category Theoretic Representations of Knotted Graphs in $\mathbb{S}^3$

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INTRODUCTION

The current work grew out of the suggestion of John Simon that there should be a monoid of "branched braids" which could play a role in the theory of knotted graphs in $\mathbb{S}^3$ analogous to that played by the Artin braid groups in classical knot theory. Rather than pursue that suggestion directly, we adopt the approach of Freyd and Yetter [2], where a succinct description of the category of tangles is given using a generalization of compact closed categories (cf. Kelly and Laplaza [8] and Joyal and Street [4]).

That work relies on classical results of Reidemeister [9] to reduce the description of ambient isotopy of knots in $\mathbb{S}^3$ to combinatorial "moves" on diagrams. In Section 1, we generalize these results to graphs embedded in $\mathbb{S}^3$. (Generalized Reidemeister moves for embedded graphs have been derived independently by L. Kauffman (cf. [6, 7]).) In Section 2, we translate this combinatorics into a description of a category of "branched tangles," and in Section 3 we use representations of the category of branched tangles to define ambient isotopy invariants of graphs embedded in $\mathbb{S}^3$.

Throughout, all spaces and maps are assumed to be piecewise linear (p.l.) except in Definition 1.1, where some subspace of p.l. spaces, homeomorphic to an open interval, are mentioned.

1. GRAPHS IN $\mathbb{S}^3$

First a few preliminaries: we begin with a rather non-standard notion of graph in which loops not incident with any vertex are permitted.

DEFINITION 1.1. A (*directed*) *graph* is a finite one complex, Γ, together with a distinguished set of points, $V(\Gamma)$, called *vertices* so that $\Gamma - V(\Gamma)$ is

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a disjoint union of circles and open intervals. The closure of components of $\Gamma - V(\Gamma)$ are called the *edges* of Γ.

An *embedded graph* is a graph, together with a p.l. homeomorphism of the graph into $\mathbb{S}^3$ (or $\mathbb{R}^3$) . We will refer to the 0-simplexes (resp. 1-simplexes) of the *image* as the "0-simplexes (resp. 1-simplexes) of the embedded graph."

DEFINITION 1.2. A *star at* $\mathbf{p}$, for $\mathbf{p}$ a point of an embedded graph, $g\colon G \to \mathbb{S}^3$, is a closed neighborhood, S, of $\mathbf{p}$ in G, which decomposes as a pointed union of 1-simplexes with $\mathbf{p}$ the common point, and such that the restriction of g to any simplex of S is linear. The point $\mathbf{p}$ is called the *center* of the star.

We say an embedded graph g' *results from a star move* on an embedded graph g if there is some star of g, S, and a homeomorphic star, S', of g' such that

$$\mathrm{Im}(g) \cap \mathrm{Im}(g') = \mathrm{Im}(g) - g(\mathrm{int}(S)) = \mathrm{Im}(g') - g'(\mathrm{int}(S'))$$

and such that there is a 2-complex, M, which is the union of a linearly embedded 1-simplex, P, with the centers of S and S' as endpoints, and linearly embedded 2-simplexes whose boundaries consist of P, and corresponding 1-simplexes from S and S'; and moreover,

$$M \cap \mathrm{Im}(g) = S$$
$$M \cap \mathrm{Im}(g') = S'.$$

We say that the 2-complex, M, *realizes* the star move. We call two embedded graphs *combinatorially equivalent* if they are equivalent under the equivalence relation generated by the relation "results from a star move on."

We now state and indicate the proof of the generalization of the classical result relating various equivalences on knots to embedded graphs:

THEOREM 1.3. *Let g and g' be embedded graphs in $\mathbb{S}^3$, then the following are equivalent*:

(1) *There is an orientation preserving homeomorphism $f\colon \mathbb{S}^3 \to \mathbb{S}^3$ such that $gf = f'$.*

(2) *There is an ambient isotopy $H\colon \mathbb{S}^2 \times I \to \mathbb{S}^3 \times I$ such that $H(g, 1) = g'$ (and, obviously, $H(g, 0) = g$).*

(3) *g and g' are combinatorially equivalent.*

Two embedded graphs related by this relation are called equivalent or isotropic.

Proof. The proof is virtually identical to the classical proof found (for instance) in Burde and Zieschang [1].

(1)$\Rightarrow$(2) and (2)$\Rightarrow$(1) depend only on properties of $\mathbb{S}^2$, not the embedded subcomplex.

For (1)$\Rightarrow$(3), the essential point to the generalization is that one can choose for embedded graphs, just as for knots, a 3-simplex outside some regular neighborhood of the subspace. The realization of a Euclidean translation of an embedded graph by star moves is essentially identical to that by Δ-moves for a knot—the only care required is to subdivide the $G \times I$ so that some subdivision of the 1-simplexes in vert$(G) \times I$ occurs as a subcomplex.

For (3)$\Rightarrow$(1), as for Δ-moves in the case of knots, it is easy to construct an orientation preserving autohomeomorphism of $\mathbb{S}^2$ which carries a star S to a star S', and fixes $\mathbb{S}^2 - U$ for U some regular neighborhood of the 2-complex M.

We now wish to move into the completely combinatorial realm of projection of embedded graphs (we now delete some point not on the graph, and pass to $\mathbb{R}^3$):

DEFINITION 1.4. A projection, p, of an embedded graph onto $\mathbb{R}^2$ is *regular* if

(1) There are only finitely many multiple points, P_i (i.e., points such that $p^{-1}(P_i)$ has more than one element).

(2) Any singular point in the curve of projection is either the image of a vertex or a double point with normal intersection but not both.

Given a Euclidean coordinate system on the plane of projection, we call a projection *normal* if it is regular, has no horizontal 1-simplexes, and all maxima, minima, crossings (double points), and vertices have distinct vertical coordinates.

As classically, a general position argument yields:

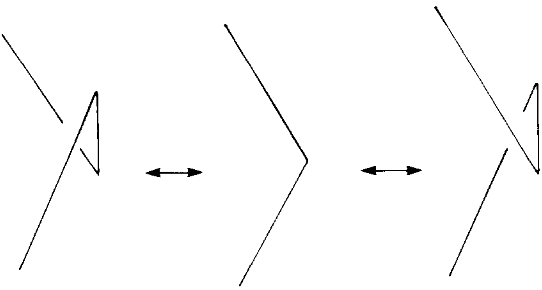
PROPOSITION 1.5. *The set of regular (resp. normal) projections is open and dense in the set of all projections.*

DEFINITION 1.6. A *diagram* is a regular projection of an embedded graph in which the overcrossing arc has been indicated at each double point. Two diagrams are *equal* if there is an isotopy of the plane carrying one to the other.

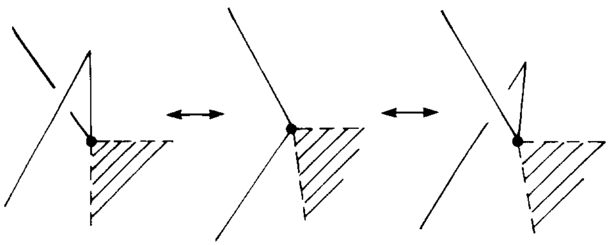
We are now in a position to give a set of "Reidemeister moves" for regular projection of embedded graphs. In the schematic renderings of the moves, a shaded region near a vertex indicates any number of edges radiating from it. An edge drawn over resp. under a shaded region overcrosses (resp. undercrosses) all edges indicated by the shaded region:

THEOREM 1.7. *Two embedded graphs are equivalent if and only if some* (*any*) *of their diagrams are related by a sequence of moves of the following forms*:

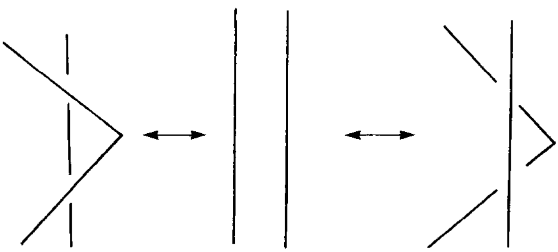
$\Omega.1$



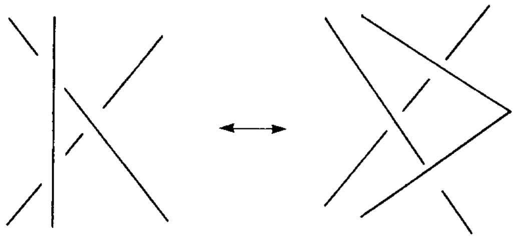
$\Omega.1V$



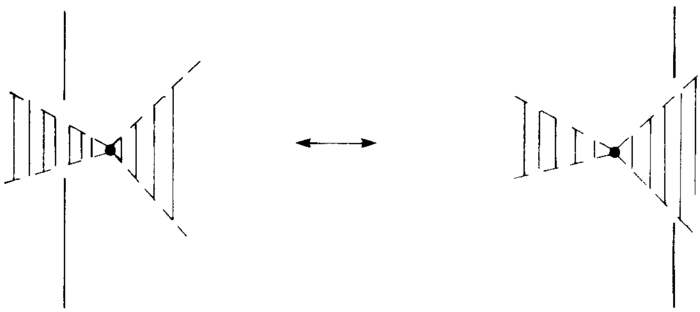
$\Omega.2$



$\Omega.3$



$\Omega.3V^-$



$\Omega.3V^+$

Proof. We must show first that any two projections of the same embedded graph are related by moves of these forms, and second that the same projections of two isotopic embedded graphs are related by moves of these forms.

For the first, consider projections as parameterized by points of $\mathbb{S}^2$. Observe that (among non-regular projections) those with triple points lie on a (possibly reducible) curve, as do those in which a double point corresponds to a vertex, those where an edge of the graph lies along a ray of projection, and those with non-normally intersecting double points. Worse "irregularities" correspond to a discrete set. Now give two projections, choose a polygonal path which avoids the discrete set of worse irregularities, and normally intersect the curves giving projections with the types of irregularities enumerated. The deformation of the projection induced by this path is then given by isotopies of the plane of projection (choose and delete a point at infinity), except when one of the curves is crossed, where, depending on the type of singularity, one of the given moves is invoked.

Now given two diagrams corresponding to two isotopic embedded graphs in some fixed projection, consider a sequence of star moves carrying one to the other. It suffices to show that any star move can be replaced by a sequence of star moves whose projections are moves of the given type or are induced by isotopies of the plane of projection.

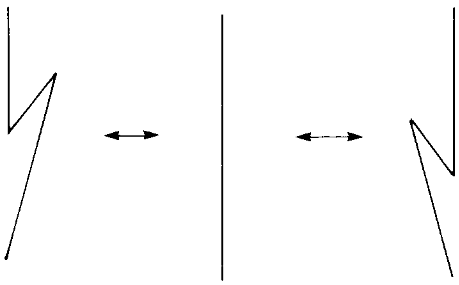
Now by subdividing the simplexes of M (the 2-complex) carrying a star move, any star move can be replaced by a sequence of star moves each involving at most one vertex of the graph, which is either the center of the star or in its boundary. By further subdivision, we can replace any star move by a sequence of star moves such that the projection M contains the projection of at most one vertex or crossing (including any vertex in the star), and if it contains no vertex or crossing such that the interior of the projection of M intersects the projection of at most one 1-simplex of the embedded graph. Case analysis of such star moves in which the center of the star is not a vertex shows that these moves induce either isotopies of the projection (if the projection of M does not intersect the projection of the original graph except in the star) or moves of types $\Omega.1$ or $\Omega.1V$ (if a 1-simplex incident with the boundary of the star intersects the interior of the projection of M), $\Omega.2$ (if a vertex-free portion of a 1-simplex intersects the interior of the projection of M), $\Omega.3$ (or other moves equivalent to $\Omega.3$ in the presence of $\Omega.2$) (if a crossing lies in the interior of the projection of M), or $\Omega.3V^+$ or $\Omega.3V^-$ (if a vertex lies in the interior of the projection of M).

Now for star moves with a vertex as center, if the projection of the 1-simplex joining the centers of the two stars intersects the projection of a vertex or crossing, use star moves inducing isotopies of the projection to move the offending vertex or crossing. The effect of this star move on the projection is then to induce an isotopy of the projection, or is the same as a move of type $\Omega.3V^+$ or $\Omega.3V^-$ followed by an isotopy of the projection.

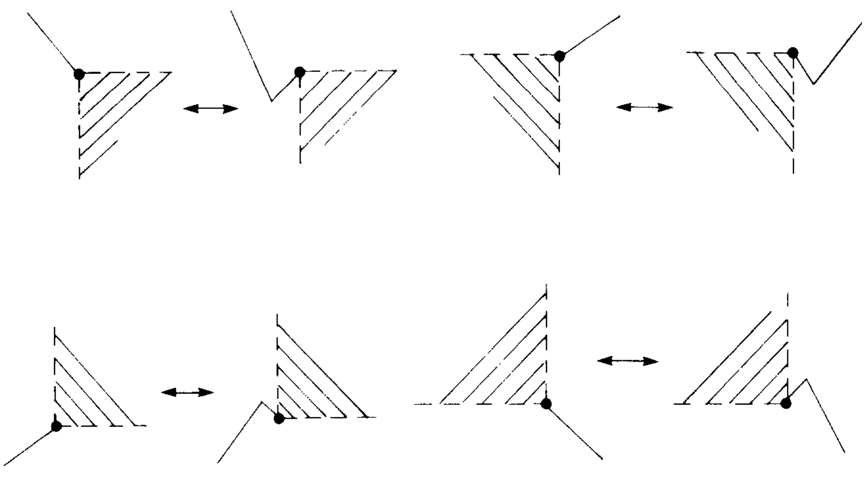
To analyse the structure of composition in the category we will consider, it is necessary, as was done in Yetter [10], to describe those isotopies of the projection between normal projections which affect the sequence of vertical coordinates of maxima, minima, crossings, and (in the new context) vertices in the diagram. As in Yetter [10] we are concerned chiefly with images of star moves which change the nature of the projection locally—those which simply interchange the order of maxima, minima, crossings, and vertices will introduce no relations not already implied by the monoidal category structure.

PROPOSITION 1.8. *Any isotopy of the projection decomposes as a sequence of isotopies each of which either does not change the order of maxima, minima, crossings, and vertices, merely interchanges two of the features in the orderings, is equivalent to a move of type $\Omega.3V^+$ or $\Omega.3V^-$ in which the vertex is of valence 2, or is of one of the following types*:

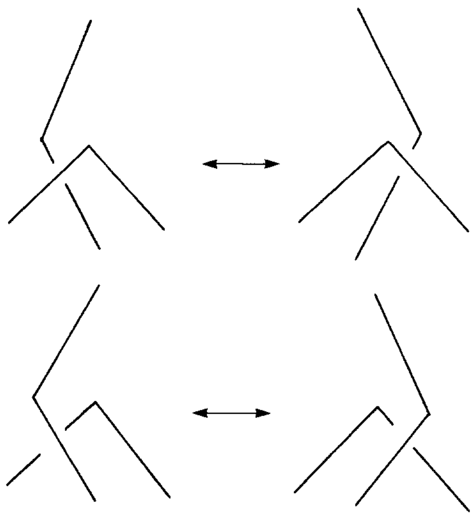
$\Delta.\pi.1$



$\Delta.\pi.1V$



$\Delta.\pi.2$



Proof. A case analysis similar to that in Yetter [10] suffices, and is left to the reader. The only point not analogous to those considered in [10] is the analysis of images of star moves which change the vertical coordinate of a vertex, thereby introducing maxima and minima: these decompose as a sequence of moves of type $\Delta.\pi.1V$.

2. Branched Tangles

We recall some category theoretic notions (see also [2–4, 8]).

DEFINITION 2.1. A *monoidal category* $\mathbb{V} = (\mathbb{V}, \otimes, I, \alpha, \rho, \lambda)$ consists of a category $\mathbb{V}$, a functor $\otimes$: $\mathbb{V} \times \mathbb{V} \to \mathbb{V}$ (written in infix notation), and natural isomorphisms $\alpha_{A,B,C}$: $(A \otimes B) \otimes C \to A \otimes (B \otimes C)$, ρ_A: $A \otimes I \to A$, and λ_A: $I \otimes A \to A$ such that

$M1$

$$\begin{tikzcd}
 & & (A \otimes B) \otimes (C \otimes D) \arrow[drr, "\alpha"] & & \\
((A \otimes B) \otimes C) \otimes D \arrow[urr, "\alpha"] \arrow[dr, "\alpha \otimes D"'] & & & & A \otimes (B \otimes (C \otimes D)) \\
 & (A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[ur, "A \otimes \alpha"'] &
\end{tikzcd}$$

and $M2$

$$\begin{tikzcd}
(A\otimes I)\otimes B \arrow[rr, "\alpha"] \arrow[dr, "\rho\otimes B"'] & & A\otimes(I\otimes B) \arrow[dl, "A\otimes\lambda"] \\
& A\otimes B &
\end{tikzcd}$$

$\mathbb{V}$ is *strict* if all components of α, ρ, and λ are identity maps.

As in Freyd and Yetter [2], much of the following will be simplified by the fact that the monoidal categories we consider are strict, and by the well-known coherence theorem of MacLane, by which any monoidal category is equivalent to a strict monoidal category. That is, one may safely assume that all monoidal categories are strict.

Definitions 2.2 and 2.3 are essentially due to Joyal and Street [4], but in Definition 2.3, we follow [2] and partially strictify the structure, since this simplifies the application to topology.

DEFINITION 2.2. A *braiding* in a monoidal category is a natural isomorphism

$$\sigma_{A,B}: A\otimes B \to B\otimes A$$

satisfying

$B1$

$$\begin{tikzcd}
& A\otimes(B\otimes C) \arrow[r, "\sigma"] & (B\otimes C)\otimes A \arrow[dr, "\alpha"] & \\
(A\otimes B)\otimes C \arrow[ur, "\alpha"] \arrow[dr, "\sigma\otimes C"'] & & & B\otimes(C\otimes A) \\
& (B\otimes C)\otimes C \arrow[r, "\alpha"'] & B\otimes(A\otimes C) \arrow[ur, "B\otimes\sigma"'] &
\end{tikzcd}$$

and $B2$

$$\begin{tikzcd}
& (A\otimes B)\otimes C \arrow[r, "\sigma"] & C\otimes(A\otimes B) \arrow[dr, "\alpha^{-1}"] & \\
A\otimes(B\otimes C) \arrow[ur, "\alpha^{-1}"] \arrow[dr, "A\otimes\sigma"'] & & & (C\otimes A)\otimes B \\
& A\otimes(C\otimes B) \arrow[r, "\alpha^{-1}"'] & (A\otimes C)\otimes B \arrow[ur, "\sigma\otimes B"'] &
\end{tikzcd}$$

DEFINITION 2.3. A (*strict*) *pivotal category* is a monoidal category, $\mathbb{V}$, equipped with a (strict) anti-involution of monoidal categories $(-)^*$ (i.e., a contravariant functor, satisfying moreover $(f\otimes g)^* = g^*\otimes f^*$, $I = I^*$, and $(-)^{**} = \mathrm{Id}_{\mathbb{V}}$), and a family of maps

$$\varepsilon_A : A\otimes A^* \to I$$

satisfying

*P*1

$$\begin{tikzcd}
(A\otimes B)\otimes(B^*\otimes A^*) \arrow[r, "\alpha"] \arrow[dddd, equal] & A\otimes(B\otimes(B^*\otimes A^*)) \arrow[r, "A\otimes\alpha^{-1}"] & A\otimes((B\otimes B^*)\otimes A^*) \arrow[d, "A\otimes(\varepsilon_B\otimes A^*)"] \\
 & & A\otimes(I\otimes A^*) \arrow[d, "A\otimes\lambda"] \\
 & & A\otimes A^* \arrow[d, "\varepsilon_A"] \\
 & & \\
(A\otimes B)\otimes(A\otimes B)^* \arrow[rr, "\varepsilon_{A\otimes B}"'] & & I
\end{tikzcd}$$

*P*2

$$\begin{tikzcd}
I\otimes I^* \arrow[rr, equal] \arrow[dr, "\varepsilon_I"'] & & I\otimes I \arrow[dl, "\lambda"] \\
 & I &
\end{tikzcd}$$

and, letting $\eta_A = (\varepsilon_A)^*$,

*P*3

$$\begin{tikzcd}
 & & (A^*\otimes A)\otimes B^* \arrow[r, "(A^*\oplus f)\otimes B^*"'] & (A^*\otimes B)\otimes B^* \arrow[r, "\alpha"'] & A^*\otimes(B\otimes B^*) \arrow[dr, "A^*\otimes\varepsilon_B"] & & \\
 & I\otimes B^* \arrow[ur, "\eta_A\otimes B^*"] & & & & A^*\oplus I \arrow[dr, "\rho"] & \\
B^* \arrow[ur, "\lambda^{-1}"] \arrow[rrrrrr, "f^*"] \arrow[dr, "\rho^{-1}"'] & & & & & & A^* \\
 & B^*\otimes I \arrow[dr, "\beta^*\otimes\eta_{A^*}"'] & & & & I\otimes A^* \arrow[ur, "\lambda"'] & \\
 & & B^*\otimes(A\otimes A^*) \arrow[r, "B^*\otimes(f\otimes A^*)"] & B^*\otimes(B\otimes A^*) \arrow[r, "\alpha^{-1}"] & (B^*\otimes B)\otimes A^* \arrow[ur, "\varepsilon_{B^*}\otimes A^*"'] & &
\end{tikzcd}$$

The new ingredient which ingredient which allows us to model embedded graphs in the categorical context is given by

DEFINITION 2.4. A *system of vertices* in a braided pivotal category is a family of maps, $v_{A,B}: A \to B$, indexed by pairs of objects, and satisfying

$V1.1$: $\sigma_{A,B} v_{A \otimes B, C} = v_{B \otimes A, C}$,

$V2.1$: $(A \otimes \eta_B)(v_{A \otimes B^*, C} \otimes B) = v_{A, C \otimes B}$, and

$V2.2$: $(\eta_A \otimes B)(A^* \otimes v_{A \otimes B, C}) = v_{B, A^* \otimes C}$.

A *graphical category* is a braided pivotal category together with a system of vertices; a *graphical functor* is a functor preserving the braiding, the pivotal structure, and the system of vertices.

PROPOSITION 2.5. *If v is a system of vertices in a braided pivotal category, then v satisfies*

$V1.2$: $\sigma^{-1}_{A,B} v_{A \otimes B, C} = v_{B \otimes A, C}$,

$V1.3$: $v_{A, B \otimes C} \sigma_{B,C} = v_{A, C \otimes B}$,

$V1.4$: $v_{A, B \otimes C} \sigma^{-1}_{B,C} = v_{A, C \otimes B}$,

$V2.3$: $(v_{A, B \otimes C} \otimes C^*)(B \otimes \varepsilon_C) = v_{A \otimes C^*, B}$, *and*

$V2.4$: $(B \otimes v_{A, B^* \otimes C})(\varepsilon_B \otimes C) = v_{B \otimes A, C}$.

Proof. $V1.2$ follows from $V1.1$ and the invertibility of σ, as does $V1.4$ from $V1.3$. $V1.3$ follows from $V1.1$, $V2.1$, and $V2.4$; while $V2.3$ (resp. $V2.4$) follows from $CC2$ and $V2.1$ (resp. $V2.2$).

We are now in a position to consider a categorical encoding of embedded graphs:

DEFINITION 2.6. A *branched tangle is* a portion of a diagram (of an embedded graph) contained in a rectangle, having no vertices on the boundary of the rectangle, and incident with the boundary only on the top and bottom edges, where it intersects transversally. Two branched tangles are *equal* if there is an isotopy of the plane carrying one to the other in such a way that corresponding sides of the boundary are preserved setwise.

DEFINITION 2.7. The *category of branched tangles*, $\mathbb{BT}$ (resp. the *category of regular branched tangles*, $\mathbb{RBT}$), has a maps all equivalence classes of tangles under the equivalence relation generated by equality, and all instances of the moves $\Omega.1$, $\Omega.1V$, $\Omega.2$, $\Omega.3$, $\Omega.3V^+$, $\Omega.3V^-$, $\Delta.\pi.1$, $\Delta.\pi.1V$, and $\Delta.\pi.2$ (resp. the same with $\Omega.1$ omitted). Composition is given by matching tops and bottoms of tangles with corresponding sequences of local orientations on the boundary. Identity morphisms are those with no maxima, minima, crossings, or vertices.

THEOREM 2.8. *$\mathbb{BT}$ (resp. $\mathbb{RBT}$) is a graphical category, when equipped with the monoidal structure, braiding, and pivotal structure described identically to those given for $\mathbb{OTang}$ in Freyd and Yetter [2], and a system of vertices given by those maps represented by branched tangles consisting of a single star at a vertex. Moreover, $\mathbb{RBT}$ is the free graphical category on one object generation, and $\mathbb{BT}$ is the free graphical category on one object generator, X, modulo the pair of relations*

$$BT1\colon \quad (X \otimes \eta_{X^*})(\sigma_{X,X} \otimes X^*)(X \otimes \varepsilon_X) = 1_X \text{ and}$$

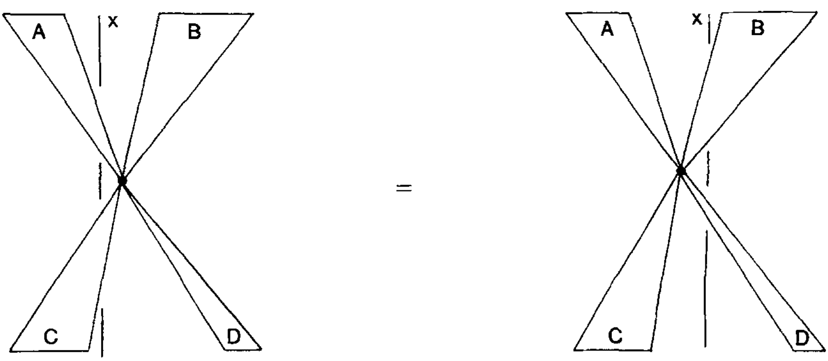
$$BT2\colon \quad (X \otimes \eta_{X^*})(\sigma_{X,X}^{1} \otimes X^*)(X \otimes \varepsilon_X) = 1_X.$$

Proof. The braided pivotal axioms are verified, as was done for (unbranched) tangles in Freyd and Yetter [2]. For the axioms for a system of vertices, note that $V1.1$ follows from $\Omega.1V$, while $V2.1$ and $V2.2$ follow from instances of $\Delta.\pi.1V$.

As in [2] the proof of freeness breaks down into three stages. First note that $\mathbb{RBT}$ is generated as a monoidal category with an anti-involution on the monoid of objects by the object X, the vertices, and the one strand components of the braiding and the pivotal structure. Thus there can be at most one functor preserving these parts of the structure from $\mathbb{RBT}$ to any graphical category, $\mathbb{C}$, and carrying the generator to a given object, A.

Next, it is shown that all instances of $\Omega.1V$, $\Omega.2$, $\Omega.3$, $\Omega.3V^+$, $\Omega.3V$, $\Delta.\pi.1$, $\Delta.\pi.1V$, and $\Delta.\pi.2$ when translated into categorical language follow from the axioms for a graphical category. For those moves not involving vertices this is already done in [2], while $\Omega.1V$ follows from $V1.1$, $\Delta.\pi.1V$ follows from $V2.1$ and $V2.2$, and $\Omega.3V^+$ and $\Omega.3V$ and $\Omega.3V^-$ follow from $B1$ and $B2$ and the invertibility and naturality of the braiding.

As an example consider $\Omega.3V^-$. To give this in categorical language, we must distinguish between those edges incident with the vertex and above it, and those incident but below: schematically we have



The following commutative diagram then verifies the equality:

$$\begin{tikzcd}
A \otimes X \otimes B \arrow[r, "\sigma \otimes B"] \arrow[d, "A \otimes \sigma^{-1}"'] \arrow[dr, phantom, "B2, \text{inv.}"] & X \otimes A \otimes B \arrow[d, "X \otimes r"] \\
A \otimes B \otimes X \arrow[ur, "\sigma"'] \arrow[d, "r \otimes X"'] \arrow[r, phantom, "\text{nat.}"] & X \otimes C \otimes D \arrow[d, "\sigma^{-1} \otimes D"] \\
C \otimes D \otimes X \arrow[ur, "\sigma"'] \arrow[r, "C \otimes \sigma"'] \arrow[ur, phantom, "B2, \text{inv.}", shift right=6] & C \otimes X \otimes D
\end{tikzcd}$$

Thus there exists a unique functor, $F\colon \mathbb{FBT} \to \mathbb{C}$, preserving the parts of the graphical structure listed in the previous paragraph and carrying the generator to A.

Finally, it is necessary to show that F preserves all components of the braiding and pivotal structure, and $(-)^*$ as an anti-involution of monoidal categories. Both are done as in [2], the first by induction and the second by using $P3$ in both categories.

Before using this characterization to obtain invariants of embedded graphs, recall that any pivotal category is equipped with a "trace":

DEFINITION 2.9. The *trace* of an endomorphism, $f\colon A \to A$, in a pivotal category is the endomorphism of I (the identity for the monoidal structure) given by

$$\mathrm{tr}(f) = \eta_{A^*}(f \otimes A^*)\varepsilon_A.$$

This trace satisfies the usual property of (linear) traces—$\mathrm{tr}(fg) = \mathrm{tr}(gf)$ whenever both composites are defined—and is a trace in the usual sense, valued in the ring $\mathrm{End}(I)$, whenever the category in question is also additive.

In the case of $\mathbb{BT}$ the "trace" of an endomorphism is the embedded graph obtained by "closing" the branched tangle in a way analogous to closing a braid (or tangle) to obtain a link. Since the trace is preserved by pivotal (and hence graphical) functors, it is possible to find invariants of embedded graphs by finding a graphical category and an object, X, therein which satisfies $BT1$ and $BT2$.

3. REPRESENTATIONS OF GRAPHICAL CATEGORIES

As in Freyd and Yetter [2], we have one primary source for categories with the requisite structure:

DEFINITION 3.1. For any (finite) group, G, a *(finite) crossed G-set* is a right G-set $\alpha\colon X\times G\to X$ equipped with a map $||\colon X\to G$ which is G-equivariant when G is regarded as acting on itself by right conjugation.

DEFINITION 3.2. The category $\mathbb{XM}_R(G)$ has as objects all crossed G-sets, with $\mathrm{Hom}(X, Y)$ being the set of all matrices, M, with rows indexed by X, and columns indexed by Y, having entries in the commutative ring R, and satisfying

$$XM1\colon\quad \forall\gamma\in G\ M_{x\gamma,\,y\gamma}=M_{x,\,y} \text{ and}$$
$$XM2\colon\quad M_{x,\,y}=\delta_{|x|,\,|y|}M_{x,\,y}.$$

As was shown in Freyd and Yetter [2], $\mathbb{XM}_R(G)$ is braided pivotal with $X\otimes Y$ having as underlying G-set $X\times Y$, with $|(x, y)|=|x|\,|y|$, and $\sigma_{X,Y}$ being the matrix of the map of crossed G-sets given by

$$(x, y)\to(y, \alpha(x, |y|)),$$

and X^* having the same underlying G-set as X, but $||$ replaced by $||^{-1}$, and

$$\eta_{X(1,x,x')}=\delta_{x,x'}(x, x')$$
$$\varepsilon_{X(x,x',1)}=\delta_{x,x'}.$$

We thus wish to find a system of vertices in $\mathbb{XM}_R(G)$:

PROPOSITION 3.3. *The matrices $v_{X,Y}$ given by*

$$v_{X,Y(x,y)}=\delta_{|x|,|y|}$$

form a system of vertices for $\mathbb{XM}_R(G)$.

Proof. It is clear that these matrices satisfy $XM1$ and $XM2$. The verification of $V1.1$, $V2.1$, and $V2.2$ are all routine calculations.

Finally, note that for an object, X, of $\mathbb{XM}_R(G)$ to satisfy $BT1$ and $BT2$ is equivalent to the condition that the "self-action"

$$x\to\alpha(x, |x|)$$

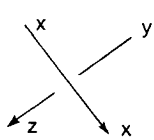
be the identity.

Thus we can find an invariant of embedded graphs by fixing a finite group, G, and a finite crossed G-set with trivial self-action, X; passing to the image under the graphical functor Φ_X: $\mathbb{RBT}\to\mathbb{XM}_R(G)$, taking the generator to X, and (noting that Φ_X factors through $\mathbb{BT}$) taking traces of the resulting maps in $\mathbb{XM}_R(G)$, we will denote the resulting invariant for the graph Γ by $\{R; X\}(\Gamma)$ (G being suppressed since it is implicit in the structure of X).

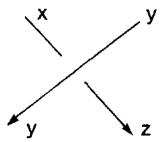
In the special case of $\{\mathbb{Z}; G\}$, where G is a crossed G-set under right conjugation with $|| = \mathrm{Id}_G$, we have as was shown in the case of links in [2]:

PROPOSITION 3.4. *For any finite group, G, and any embedded graph, Γ, $\{\mathbb{Z}; G\}\}(\Gamma)$ is the number of group homomorphisms from $\pi_1(\mathbb{S}^3 - \Gamma)$ to G.*

Proof. Given any diagram of an embedded graph, we can obtain a presentation for the fundamental group of the complement in a way analogous to the Wirtenger presentation for the fundamental group of a link complement: take as generators the arcs of the projection, at crossings take relations of the forms

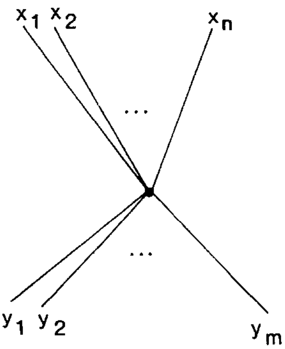


$z = x^{-1} y x$



$z = y x y^{-1}$

and at vertices of the form



$$x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} = y_1^{\eta_1} y_2^{\eta_2} \cdots y_m^{\eta_m}$$

$$\epsilon_i, \eta_j = \begin{cases} 1 & \text{if edge oriented downward} \\ -1 & \text{if edge oriented upward} \end{cases}$$

Routine considerations show that the set of generators can be reduced to those arcs, say N in number, containing maxima of the projection or having a vertex as a maximum (in terms of the vertical coordinate), and the relations can be reduced to those arising from the vertices by equating different possible names (in terms of this smaller set of generators) for arcs containing minima.

It is plain from the standard method of calculating $\{\mathbb{Z}; G\}$ from a diagram that we are precisely counting the number of N-tuples of elements in G, which satisfy these relations—that is, the number of group homomorphisms from $\pi_1(\mathbb{S}^3 - \Gamma)$ to G.

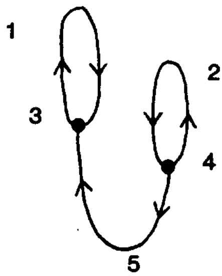
It is not clear whether there is a suitable generalization of Joyce's knot quandle (see [5]) to the case of embedded graphs which would give an interpretation of more general $\{\mathbb{Z}; X\}$'s; nor is it clear whether any normalization procedure can be laid down so that invariants extending those knot invariants obtained by evaluating an invariant of framed links on the 0-framing to the larger context can be found (cf. Freyd and Yetter [2]).

We conclude with sample calculations, and a small table of values for some invariants of the form $\{R; X\}$ on some simple embedded graphs.

Sample Calculations

We calculate $\{\mathbb{Z}; X\}$ for a generic crossed G-set, X. Throughout, we regard the underlying sets of X and X^* as the same, and distinguish occurrences of elements of X when regarded as elements of X^* by underlining. We represent the action of the matrices on free modules over various powers of X with basis given by n-tuples of elements of X. Each line of the calculation corresponds to the "feature" (maximum, minimum, crossing, or vertex) of the diagram with the corresponding number. Summation indices not specified as lying in some other set range over the crossed G-set, X.

EXAMPLE 1.



1 $\quad 1 \to \sum_{x} (\underline{x}, x)$

2 $\quad \to \sum_{x, y} (\underline{x}, x, y, \underline{y})$

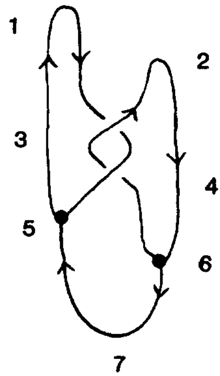
3 $\quad \to \sum_{x, y} \sum_{z \in \{z \mid |z| = 1\}} (\underline{z}, \underline{y}, y)$

4 $\quad \to \sum_{x, y} \sum_{z \in \{z \mid |z| = 1\}} \sum_{w \in \{w \mid |w| = 1\}} \{\underline{z}, w)$

5 $\quad \to \sum_{x, y} \sum_{z \in \{z \mid |z| = 1\}} \sum_{w \in \{w \mid |w| = 1\}} \delta_{z, w}$

$$= (\# X)^2 (\# \{z \mid |z| = 1\})$$

EXAMPLE 2.



1 $$1 \to \sum_{x} (\underline{x}, x)$$

2 $$\to \sum_{x,y} (\underline{x}, x, y, \underline{y})$$

3 $$\to \sum_{x,y} (\underline{x}, y, \alpha(x, |y|), \underline{y})$$

4 $$\to \sum_{x,y} (\underline{x}, \alpha(x, |y|), \alpha(y, |y|^{-1}|xx||y|), \underline{y})$$

5 $$\to \sum_{x,y} \sum_{z \in \{z \mid |z| = |y|^{-1}|x|^{-1}|y||x|\}} (\underline{z}, \alpha(y, |y|^{-1}|x||y|), \underline{y})$$

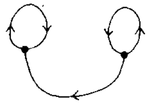
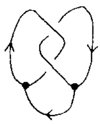
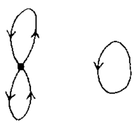
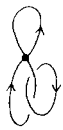
6 $$\to \sum_{x,y} \sum_{z \in \{z \mid |z| = |y|^{-1}|x|^{-1}|y||x|\}} \sum_{w \in \{w \mid |w| = |y|^{-1}|x|^{-1}|y||x|\}} (\underline{z}, w)$$

7 $$\to \sum_{x,y} \sum_{z \in \{z \mid |z| = |y|^{-1}|x|^{-1}|y||x|\}} \sum_{w \in \{w \mid |w| = |y|^{-1}|x|^{-1}|y||x|\}} \delta_{z,w}$$

$$= \sum_{x,y} \#\{z \mid |z| = [|y|^{-1}, |x|^{-1}]\}$$

If, for example, X is the crossed $\sum_3$-set with underlying set $\sum_3 + 1$, $\sum_3$ acting on the first summand by right conjugation, and on the second trivially, with $||$ being the obvious map, then $\{\mathbb{Z}, X\}$ of the first graph above is 98, while that of the second graph is 80.

In the following table, X is the crossed $\sum_3$-set described above, and $\sum_3$ is the symmetric group on three letters with the obvious crossed $\sum_3$-set structure.

Γ	$\{\mathbb{Z}, X\}(\Gamma)$	$\{\mathbb{Z}, \Sigma_3\}(\Gamma)$
	98	36
	80	36
	58	36
	49	36
	343	216
	217	108
	108	90

Acknowledgments

The author thanks the National Science Foundation for support while he was in residence at the Institute for Advanced Study, where this work was begun (Grant DMS-8610730(1)), and the Groupe Interuniversitaire en Études Catégoriques for support while this work was completed. The author also thanks the referee for pointing out an error in an earlier version of this work.

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